

Upper-critical graphs (complete k -partite graphs)

José Antonio Martín H.*

Faculty of Computer Science, Complutense University of Madrid, Spain

Abstract

This work introduces the concept of *upper-critical graphs*, in a complementary way of the conventional (lower)critical graphs: an element x of a graph G is called *critical* if $\chi(G - x) < \chi(G)$. It is said that G is a *critical graph* if every element (vertex or edge) of G is critical. Analogously, a graph G is called *upper-critical* if there is no edge that can be added to G such that G preserves its chromatic number, i.e. $\{ e \in E(\overline{G}) \mid \chi(G + e) = \chi(G) \} = \emptyset$. We show that the class of upper-critical graphs is the same as the class of complete k -partite graphs. A characterization in terms of hereditary properties under some transformations, e.g. subgraphs and minors and in terms of construction and counting is given.

Key words: Graph-coloring; critical-graphs; upper-critical graphs; complete k -partite graphs

*address: Facultad de Informática, Universidad Complutense de Madrid, C. Prof. José García Santesmases, s/n., 28040, Madrid, (Spain). Tel.: +34 91.394.7656, Fax: +34 91.394.7510

Email address: jamartinh@fdi.ucm.es (José Antonio Martín H.)

1. Preliminary definitions and basic terminology

Unless we state it otherwise, all graphs in this work are connected and simple (finite, and have no loops or parallel edges).

Partitioning the set of vertices $V(G)$ of a graph G into separate classes, in such a way that no two adjacent vertices are grouped into the same class, is called the vertex coloring problem. In order to distinguish such classes, a set of colors C is used, and the division into these (*color*)-*classes* is given by a coloring $c : V(G) \rightarrow \{1 \dots k\}$, where $c(x) \neq c(y)$ for all xy 's belonging to the set of edges $E(G)$ of G , where $c^*(x)$ is the color-class of x (i.e. the color-class that contains vertex x).

Given a graph coloring over a graph G with a set of colors C , if C has cardinality k , then c is a k -*coloring* of G . The *chromatic number* of a graph $\chi(G)$ is the minimum number of different colors which are necessary for coloring the vertices of a graph G . If $\chi(G) \leq k$ then G is k -*colorable* (i.e. can be colored with k different colors) and if $\chi(G) = k$ then G is k -*chromatic*. A graph is *complete* if every two distinct vertices in the graph are adjacent. The complete graph of order k is denoted by K_k .

The *join* $G_1 + G_2$ of two different (vertex-disjoint) graphs, G_1 and G_2 , is the graph that has vertex set $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1), v \in V(G_2)\}$, that is, each vertex of G_1 joined to each vertex of G_2 .

An *independent set* (also called *stable set*) I is a set of vertices of G such that there are no edges between any two vertices in I . The *complement* of G is denoted by \overline{G} , e.g. An independent set I of G is a subgraph of \overline{G} which is complete (i.e. a *complete subgraph* or *clique* of \overline{G}), therefore, we can denote

an independent set of cardinality n as \overline{K}_n .

A *complete k -partite graph* K_{n_1, n_2, \dots, n_k} is a graph isomorphic to the join of k independent sets of cardinality n_1, n_2, \dots, n_k respectively, i.e., $K_{n_1, n_2, \dots, n_k} = \overline{K}_{n_1} + \overline{K}_{n_2} + \dots + \overline{K}_{n_k}$, e.g. $K_{2,2}$ is a square.

The set of all adjacent vertices to a vertex $x \in V(G)$ is called its *neighborhood* and is denoted by $N_G(x)$. When it is clear to which graph we are referring to, we will use simply $N(x)$, omitting the graph. The *closed neighborhood* of a vertex x , denoted by $N[G]$, includes also the vertex x , i.e. $N[x] = N(x) \cup \{x\}$. If $N(x) = N(y)$ we say that vertex y is a copy of x and vice versa.

The degree of a vertex x , $\deg(x)$, is equal to the cardinality of its neighborhood $\deg(x) = |N(x)|$. A *complete vertex* is any $x \in V(G)$ such that $N[x] = V(G)$ and hence if every vertex is complete the graph is also complete K_k .

An *edge contraction* denoted by G/xy or G/e is the process of replacing two adjacent vertices x, y of G , i.e. $xy \in E(G)$, by a new vertex z such that $N(z) = N(x) \cup N(y)$.

A graph H is called a *minor* of the graph G ($H \prec G$) if H is isomorphic to a graph that can be obtained from a subgraph of G by zero or more edge deletions, edge contractions or vertex deletions on a subgraph of G . In particular, G is minor of itself.

A *vertex identification* denoted by $G/x, y$ is the process of replacing two non-adjacent vertices x, y of G , i.e. $xy \notin E(G)$, by a new vertex z such that $N(z) = N(x) \cup N(y)$.

A graph $H = G/S$ is called a *contraction* of the graph G if H is isomorphic

to a graph that can be obtained by zero or more vertex identifications on G , where S is the set of vertex identifications to obtain H . In particular, G is a contraction of itself.

When S induces a contraction H of G such that $H = K_k$, H is called a *collapse* of G . Thus, since a collapse $H = K_k$ is also a partition of all the vertices of G in k independent sets, $\chi(G)$ is the size of the minimum collapse of G and if $k = \chi(G)$ then $H = K_k$ induces a k -coloring of G .

Critical graphs were first studied by Dirac [1, 2, 3]. An element x of a graph G is called *critical* if $\chi(G - x) < \chi(G)$. If all the vertices of a graph G are critical we say that G is *vertex-critical* and if every element (vertex or edge) of G is critical we say that G is a *critical graph*, specifically, if $\chi(G) = k$, G is k -critical.

Examples of critical graphs in general are the *complete graphs* K_k of size $|V(G)| = k$, odd cycles are the only 3-critical graphs and odd wheels are just one case of 4-critical graphs. If x is a critical vertex of a graph G then x is a color-class itself, that is, there is at least one k -coloring of G where x is the only vertex with the $c(x)$ color.

Also k -critical graphs possess the next well known properties:

- G has only one component.
- G is finite
- Every vertex is adjacent to at least $k-1$ others.
- If G is $(k-1)$ -regular, meaning every vertex is adjacent to exactly $k-1$ others, then G is either K_k or an odd cycle.

- $|V(G)| \neq k + 1$
- If G is different from K_k then $|V(G)| \geq k + 2$.

2. Upper-critical graphs

This work introduces the concept of *upper-critical graphs*, in a complementary way of the conventional (lower)critical graphs. Also, we show that a k -chromatic graph G is upper-critical if and only if it is isomorphic to a complete k -partite graph, showing that the class of the upper-critical graphs coincides with the class of the complete k -partite graphs.

Definition 1. *A graph G is called upper-critical if there is no edge (e) that can be added to G such that G preserves its chromatic number, i.e.*

$$\{e \in E(\overline{G}) \mid \chi(G + e) = \chi(G)\} = \emptyset$$

or alternatively,

$$\chi(G + e) > \chi(G) \quad \forall e \in E(\overline{G}), \quad \text{or} \quad G = K_k,$$

If $\chi(G) = k$, G is called a k -chromatic upper-critical graph.

2.1. General structural properties

Theorem 1. *If G is a k -chromatic upper-critical graph then every k -coloring of G induces the same partition of vertices of G , in k different color classes, i.e. G is uniquely colorable (uniquely k -colorable, to be more precise).*

Proof. Given a k -chromatic upper-critical graph G , i.e. $\chi(G+e) > \chi(G)$. Let us suppose that there are at least two different colorings of G that induce

two different partitions of G . So, there is at least a vertex x that can be assigned to either a color-class c_1 or a color-class c_2 . Hence, it is possible to add a new edge $e = xy$ to G ($G + e$), from vertex x towards an element of c_2 (say vertex y). But, in this case, $G + e$ has at least one k -coloring which is a contradiction since G is upper-critical. \square

Theorem 2. *If G is an upper-critical graph, x a vertex of G and c a k -coloring of G then:*

$$N(x) = V(G) - \{y \in V(G) \mid c(x) \neq c(y)\}$$

Proof. Suppose $N(x) \neq V(G) - \{y \in V(G) \mid c(x) \neq c(y)\}$. Then either c is not a k -coloring of G or there is a vertex $y \in V(G)$ such that $y \notin N(x)$ and $c(x) \neq c(y)$. Hence $\chi(G + xy) = \chi(G)$, which is a contradiction since G is upper-critical. \square

Theorem 3. *If G is a k -chromatic upper-critical graph then:*

$$K_k \subseteq G$$

Proof. Since G has k color classes, let $x_1, x_2, x_3, \dots, x_k$ be k vertices such that each x_i belongs to a different color class c_i . Now, the induced subgraph $x_1, x_2, x_3, \dots, x_k$ is a complete graph K_k since, by theorem 2, $x_i x_j \in E(G) \forall i \neq j$. \square

2.2. Hereditary properties of transformations, subgraphs and minors

In this section we will proof that upper-critical graphs are closed under vertex deletion, vertex identification and edge contraction. And also, for particular cases, under vertex addition, edge addition and edge deletion, i.e., that given an upper-critical graph G , the next graphs are also upper-critical:

1. $G - x$.
2. $G/x, y$.
3. G/xy .
4. $G + x$; if x is a copy of some vertex of G or x is a complete vertex.
5. $G + e$; for $e = xy \notin E(G)$ if either:
 - (a) $N[x] \neq K_k$ or,
 - (b) $|E(\overline{G})| \leq |V(G)| - \chi(G)$
6. $K_k - \{e_1, \dots, e_{k-2}\}$; for some particular sequence of critical edges.

Subsequent sections, will study how are the graphs obtained from G by applying each particular graph transformation, e.g., decrease, increase or maintain its chromatic number or its order.

Theorem 4. *If G is an upper-critical graph and $xy \notin E(G)$ then:*

$$N(x) = N(y), \text{ i.e. vertex } y \text{ is a "copy" of vertex } x$$

Proof. Let us suppose there is a vertex z in $N(y)$ but not in $N(x)$, i.e. $N(x) \neq N(y)$, then:

- $\chi(G + xz) > \chi(G)$ hence $c(x) = c(z)$ for every χ -coloring c of G .
- $\chi(G + xy) > \chi(G)$ hence $c(x) = c(y)$ for every χ -coloring c of G .

Hence, $c(z) = c(y)$ for every χ -coloring c of G , which is a contradiction since $yz \in E(G)$. □

Theorem 5. *If G is an upper-critical graph and x a vertex of G then:*

$$G + y \text{ is also upper-critical if } y \text{ is a copy of } x$$

Proof. The case where G is a complete graph is trivial. Let G be an upper-critical graph different from a complete graph. Now, let $G + y$ be the graph obtained by adding a copy (y) of the vertex x to G . Then, since $c(y) = c(x)$ in every k -coloring of G then $G + y$ is k -chromatic and, since $\chi(G + y + e) > \chi(G)$ for all $e \notin E(G + y)$ then $G + y$ is upper-critical. \square

Theorem 6. *If G is an upper-critical graph and x a vertex of G then:*

$$G - x \text{ is also upper-critical}$$

Proof. Let us suppose that G is a k -chromatic upper-critical graph and e a non-existent edge in G . Assume $G - x$ is not upper-critical. We can consider just two cases for $G - x$:

1. If $\chi(G - x) = k - 1$ then $(G - x) + e$ has a $(k-1)$ -coloring (since $G - x$ is not upper-critical), hence there is a k -coloring of $(G - x) + e + x$, i.e. a k -coloring of $G + e$, which is a contradiction since G is a k -chromatic upper-critical graph.
2. If $\chi(G - x) = k$ then x is not a complete vertex, meaning that there is another vertex y such that $N(x) = N(y)$ and $xy \notin E(G)$. Now, since $G - x$ is not upper-critical there are two vertices $u, v \in V(G - x)$ such that $uv \notin E(G - x)$ and there is a k -coloring (c) of $G - x$ where $c(u) \neq c(v)$, i.e., a k -coloring of $(G - x) + e$. However, since $N(x) = N(y)$ we can restore back vertex x assigning to it the color $c(y)$ obtaining a k -coloring of G such that there are two vertices $u, v \in V(G)$, $uv \notin E(G)$ and $c(u) \neq c(v)$, i.e., we can obtain a k -coloring of $G + e$, which is a contradiction since G is upper-critical.

□

Theorem 7. *If G is an upper-critical graph and x, y are two vertices of G such that $xy \notin E(G)$ then:*

$G/x, y$ is also upper-critical

Proof. Since G is upper-critical and $xy \notin E(G)$ then, by theorem 4, vertex y is a copy of x and hence $G/x, y = G - x$. Now, since $G - x$ is upper-critical then $G/x, y$ is upper-critical. □

Theorem 8. *If G is an upper-critical graph and $xy \in E(G)$ then:*

G/xy is also upper-critical

Proof. by induction on the number $n = V(G)$ of vertices of G . Let e be some edge of G .

Base $n = 2$: The K_2 graph is upper-critical by definition and $K_2/e = K_1$ and K_1 is upper-critical by definition.

Assume true for every upper-critical graph in n vertices.

Proof for $n + 1$: Let G be an upper-critical graph on $n + 1$ vertices such that G/e is not upper-critical.

1. If $G = K_k$ then G/e is also a complete graph hence G/e is upper-critical which is a contradiction.
2. If $G \neq K_k$ then there are two cases:
 - (a) x is a complete vertex (this case includes all three cases: x , y or both). It is easy to see that $G/xy = G - x$, but $G - x$ is upper-critical, thus there is a contradiction.

- (b) Since $G \neq K_k$ and x is not a complete vertex: we can delete a copy (z) of x from G , obtaining a new graph $H = G - z$. Since G is upper-critical then H is upper-critical and thus $(G - z)/e$ is upper-critical by the inductive hypothesis. But now since z is a copy of x then:

$$(G - z)/e + z = G/e$$

is upper-critical by theorem 5 which is a contradiction.

□

Theorem 9. *If G is an upper-critical graph, x, y are two vertices of G such that $xy \notin E(G)$ then:*

$G + xy$ is also upper-critical if:

1. $N[x] \neq K_k$
2. $|E(\overline{G})| \leq |V(G)| - k$

Proof. Since $xy \notin E(G)$ there is always a vertex z , adjacent to both x and y .

1. There is at least one z which is not a complete vertex: Let \dot{z} be a copy of z . Then $G + \dot{z}$ is upper-critical and so $(G + \dot{z})/x\dot{z}$. Now:

$$(G + \dot{z})/x\dot{z} = G + xy = G + e$$

2. Every z is a complete vertex: Fix G to be k -chromatic. Then $G + e$ has K_{k+1} as a subgraph and $G + e + e_2$ will have K_{k+2} as a subgraph, for a new edge e_2 , so $\chi(G + e + e_2) > \chi(G + e)$. Hence $G + e$ is upper-critical.

Therefore $G + e$ is upper-critical. \square

Theorem 10. *If $G = K_k$ then there is always a particular sequence (including the empty sequence) of critical edges such that:*

$$G = K_k - \{e_1, \dots, e_{k-2}\} \text{ is also upper-critical}$$

Proof. Let G be a 2-chromatic (bipartite) upper-critical graph on N vertices then, by definition, $G + e$ is $(k+1)$ -chromatic and, by theorem 9, $G + e$ is also upper-critical, hence it is possible to add edges obtaining successive upper-critical graphs up to obtain a complete graph. Hence, given a particular complete graph there is always a particular sequence (including the empty sequence) of critical edges for obtaining successive upper-critical graphs with lower chromatic number up to a bipartite graph. \square

2.3. Construction, characterization and counting

A procedure to obtain an upper-critical graph G , starting from any other graph, is to find a k -coloring of some k -chromatic graph H and add the edges $x_i x_j$ to H whenever the color class of vertex x_i is different from the color class of x_j , i.e. $x_i x_j \in E(G) \forall i, j : c(x_i) \neq c(x_j)$. Thus, every coloring of any particular graph corresponds to a particular upper-critical graph.

Therefore, the total number of different colorings (excluding repeated partitions) of all graphs is equal to the the total number of upper-critical graphs.

Furthermore, there is a direct way to obtain any arbitrary upper-critical graph. Upper-critical graphs, contrary to (lower)critical-graphs, has a very easy general characterization and description, for instance, the n -vertex list

notation where $G = \{n_1, n_2, \dots, n_k\}$ is the k -chromatic upper-critical graph where the positive integers $\{n_1, n_2, \dots, n_k\}$ indicate the number of vertices in each color class (c_i) respectively.

It is immediate to see that the n -vertex list is unique in the sense that two upper-critical graphs share the same n -vertex list if and only if they are isomorphic.

From this it follows that it is possible to specify directly an arbitrary upper-critical graph using the n -vertex list notation subject to just one constraint:

$$|V(G)| = \sum_{i=1}^k n_i \quad (1)$$

where G is a k -chromatic upper-critical graph and n_i is the number of vertices belonging to the k th color class.

As we can see, the definition of an upper-critical graph is the same as the definition of a complete k -partite graph. Theorem 11 formalizes this equivalence.

Theorem 11. *A k -chromatic graph G is upper-critical if and only if it is isomorphic to a complete k -partite graph $H = K_{n_1, n_2, \dots, n_k}$.*

Proof. It is clear that H is k -chromatic and

$$\{e \in E(\overline{H}) \mid \chi(H + e) = k\} = \emptyset,$$

so H is an upper-critical graph. Also, since G is k -chromatic it can be divided, by means of a k -coloring, into k -color classes (independent sets), which, by theorem 1, induces always the same partition. Now, by theorem 2, for every $x \in V(G)$ $N(x) = V(G) - \{y \in V(G) \mid c(x) \neq c(y)\}$ meaning that

Table 1: Upper-critical graphs on $|V(G)|$ vertices vs. Chromatic number $k = \chi(G)$

| $V \setminus k$ | 1 | 2 | 3 | 4 | 5 | ... |
|-----------------|-------|----------------------|----------------------------|------------------|----------|----------|
| 1 | K_1 | | | | | |
| 2 | | K_2 | | | | |
| 3 | | $\{1, 2\}$ | K_3 | | | |
| 4 | | $\{1, 3\}, \{2, 2\}$ | $\{1, 1, 2\}$ | K_4 | | |
| 5 | | $\{1, 4\}, \{2, 3\}$ | $\{1, 1, 3\}, \{1, 2, 2\}$ | $\{1, 1, 1, 2\}$ | K_5 | |
| \vdots | | \vdots | \vdots | \vdots | \vdots | \ddots |

each element of an independent set of G is completely joined to any other element of a different independent set of G . Therefore G can be described as the join of k independent sets, $G = K_{n_1, n_2, \dots, n_k}$, i.e., G is a complete k -partite graph. \square

Therefore, the above n -vertex list notation $G = \{n_1, n_2, \dots, n_k\}$ can be replaced by the standard notation K_{n_1, n_2, \dots, n_k} .

Table 1 shows the space of upper-critical graphs in $(|V|, \chi)$ coordinates using the n -vertex list notation. A sample is shown up to $|V| = 5$ and $\chi = 5$.

Furthermore, it is possible to count the number $P(N, k)$ of upper-critical graphs (i.e. complete k -partite graphs) given the number ($N = |V|$) of vertices and the chromatic number (k). A *partition* of a positive integer N is a way to express N as the sum of positive integers [4], i.e. $4 = 2 + 2$ or $4 = 3 + 1$. Therefore, every partition of a positive integer defines a unequivocal upper-critical graph and viceversa.

Let $P(N, k)$ denote the number of ways of writing N as a sum of exactly

k terms, then $P(N, k)$ can be computed from the next recurrence relation:

$$P(N, k) = P(N - 1, k - 1) + P(N - k, k), \quad (2)$$

Let $\mathbb{S} = |V| \times \chi$ be the space of all upper-critical graphs of order $N = |V|$ and chromatic number $k = \chi$, it is possible to go from a particular point $\mathbb{S}(N_i, k_i)$ to another point $\mathbb{S}(N_j, k_j)$ in \mathbb{S} , i.e., traveling across table 1 cells. Note that to each point of \mathbb{S} could be associated several different upper-critical graphs sharing the same order and chromatic number. Theorem 10 formalizes this fact.

Theorem 12. *If G is an upper-critical graph with $N = |V(G)|$ vertices and chromatic number k then:*

1. $G + x \in$
 - (a) $\mathbb{S}(N + 1, k)$ (if x is a copy of some non-complete vertex of G)
 - (b) $\mathbb{S}(N + 1, k + 1)$ (if x is a complete vertex)
2. $G - x \in$
 - (a) $\mathbb{S}(N - 1, k)$ (if x is a non-complete vertex)
 - (b) $\mathbb{S}(N - 1, k - 1)$ (otherwise)
3. $G/x, y \in \mathbb{S}(N - 1, k)$
4. $G/xy \in$
 - (a) $\mathbb{S}(N - 1, k - 1)$ (if x and y are both complete vertices)
 - (b) $\mathbb{S}(N - 1, k)$ (if x or y are non-complete vertices)
 - (c) $\mathbb{S}(N - 1, k + 1)$ (if x and y are non-complete vertices)
5. $G + xy \in \mathbb{S}(N, k + 1)$ if either:
 - (a) $N[x] \neq K_k$ or,

$$(b) |E(\overline{G})| \leq |V(G)| - \chi(G)$$

$$6. K_k - \{e_1, \dots, e_m\} \in \mathbb{S}(N, k - m)$$

Proof.

1. ...

- (a) if x is a copy of some non-complete vertex of G then $\chi(G) = \chi(G + x)$ since x can be colored with the color of another vertex of G and $G + x$ has $N + 1$ vertices.
- (b) Any graph plus a complete vertex increase both in vertex number and chromatic number, by one.

2. ...

- (a) Since x is a non-complete vertex then it is a copy of another vertex y , so $\chi(G) = \chi(G - x)$.
- (b) Any graph minus a complete vertex decrease both in vertex number and chromatic number, by one.

3. Since x and y are non-complete vertices then $G/x, y = G - x$.

4. ...

- (a) if x and y are both complete vertices then $G/xy = G - x$
- (b) if x or y are non-complete vertices then $G/xy = G - x$
- (c) if x and y are non-complete vertices then G/xy is equivalent to remove a copy of some vertex, $G - z$, and adding some edges,

$G - z + e_1, e_2, \dots$, which by definition will increase the chromatic number.

- 5. By definition and theorem 9.
- 6. By definition and theorem 10.

□

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